

Critical exponents and dimension for generalised limit sets

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Joint work with Jonathan Fraser

Motivation: hyperbolic geometry

▶ $\mathbb{D}^n = \{z \in \mathbb{R}^n : |z| < 1\}$

▶ Hyperbolic metric d_H :

$$|ds| = \frac{2|dz|}{1 - |z|^2}$$

▶ $S^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}$

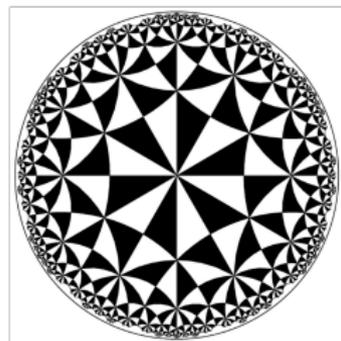
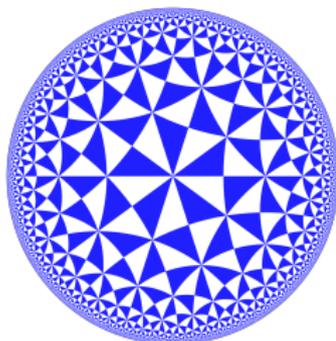
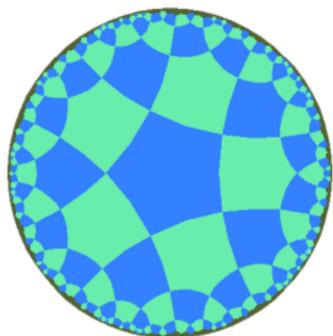
▶ $\text{Con}^+(n)$: the group of orientation preserving isometries of (\mathbb{D}^n, d_H)

▶ A subgroup $\Gamma \leq \text{Con}^+(n)$ is *Kleinian* if it is discrete

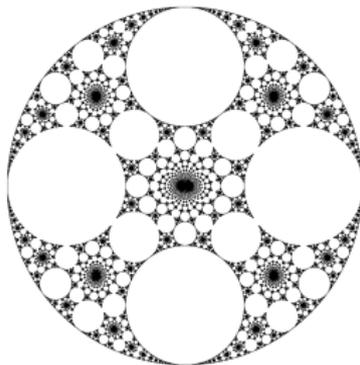
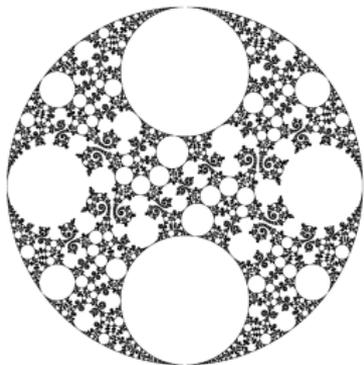
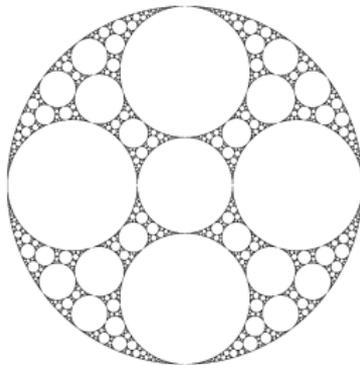
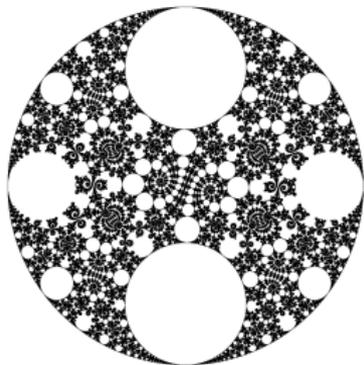
▶ The *limit set* of a Kleinian group Γ is

$$L(\Gamma) := \overline{\Gamma(0)} \setminus \Gamma(0)$$

Motivation: hyperbolic tilings



Motivation: Kleinian limit sets



Motivation: Kleinian limit sets

- ▶ The *Poincaré series* $P_\Gamma : [0, \infty) \mapsto [0, \infty]$ is given by

$$P_\Gamma(s) = \sum_{g \in \Gamma} \exp(-sd_H(0, g(0))) = \sum_{g \in \Gamma} \left(\frac{1 - |g(0)|}{1 + |g(0)|} \right)^s$$

- ▶ The *Poincaré exponent* is defined as

$$\delta'(\Gamma) = \inf\{s \geq 0 : P_\Gamma(s) < \infty\}.$$

Motivation: dimension theorem of Kleinian limit sets

Theorem (Patterson, Sullivan, Stratmann–Urbański, Bishop–Jones)

If Γ is non-elementary and geometrically finite, then

$$\dim_{\mathbb{B}} L(\Gamma) = \dim_{\mathbb{H}} L(\Gamma) = \delta'(\Gamma).$$

Question

Can we do this for general discrete subsets of \mathbb{D}^n ?

In other words - can we replace $\Gamma(0)$ with a general discrete $E \subseteq \mathbb{D}^n$?

Generalised limit sets

Let $E \subseteq \mathbb{D}^n$ be discrete.

- ▶ The *limit set* of E is

$$L(E) = \overline{E} \setminus E$$

- ▶ The associated *accumulation series* $S_E : [0, \infty) \mapsto [0, \infty]$ is given by

$$S_E(s) = \sum_{x \in E} (1 - |x|)^s$$

- ▶ We define the *critical exponent* $\delta(E)$ to be

$$\delta(E) = \inf\{s \geq 0 : S_E(s) < \infty\} = \sup\{s \geq 0 : S_E(s) = \infty\}$$

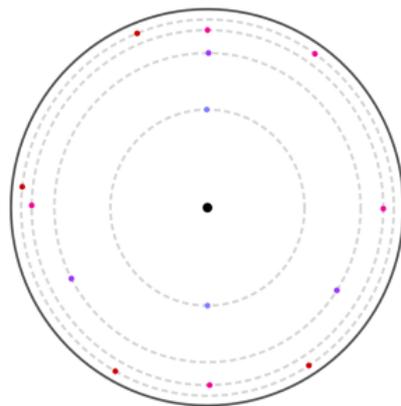
where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$

Generalised limit sets

Assumption: $L(E) \subseteq S^{n-1}$.

Generalised limit sets: examples

- 1 $E = \{x_k\}_k$, where the x_k converge along a common line to a single point $w \in S^{n-1}$ and suppose x_k is a distance $1/\log k$ from the boundary. Then $L(E) = \{w\}$, and so $\dim_{\text{H}} L(E) = \dim_{\text{B}} L(E) = 0$ but $\delta(E) = \infty$.
- 2 For each $k \in \mathbb{N}$ let $E_k \subseteq \mathbb{D}^n$ consist of k maximally separated points all of distance 2^{-k} from the boundary S^{n-1} and let $E = \cup_k E_k$. Then $L(E) = S^{n-1}$ with $\dim_{\text{H}} L(E) = \dim_{\text{B}} L(E) = n - 1$, but $\delta(E) = 0$.



Lower bound

$$\delta(E) \leq \overline{\dim}_{\mathbf{B}} L(E)?$$

Lower bound

- ① E is *well-approximated* if there exists some constant $c_1 \geq 1$ such that for all $x \in E$, there is an element $z \in L(E)$ satisfying

$$|x - z| \leq c_1(1 - |x|).$$

- ② E is *separated* if there exists a constant $c_2 \in (0, 1)$ such that for all $x \in E$,

$$B(x, c_2(1 - |x|)) \cap E = \{x\}.$$

$$\implies \delta(E) \leq \overline{\dim_{\mathbb{B}} L(E)}$$

Upper bound

The *radial limit set* of E is defined by

$$L_{rad}(E) = \{z \in S^{n-1} : \exists c \geq 1 \text{ s.t. } \forall r > 0, \exists x \in \Gamma(0) \\ \text{s.t. } |z - x| \leq c(1 - |x|) \leq cr\}.$$

Theorem

If $E \subseteq \mathbb{D}^n$ is discrete, then

$$\dim_{\mathbb{H}} L_{rad}(E) \leq \delta(E).$$

Main result

Theorem

Let $E \subseteq \mathbb{D}^n$ be discrete.

- ① If E is separated and well-approximated, then

$$\overline{\dim}_{\mathbb{B}} L(E) \geq \delta(E)$$

and

$$\dim_{\mathbb{H}} L_{rad}(E) \leq \delta(E).$$

- ② If in addition E satisfies $\dim_{\mathbb{H}} L_{rad}(E) = \overline{\dim}_{\mathbb{B}} L(E)$, then

$$\dim_{\mathbb{H}} L(E) = \dim_{\mathbb{B}} L(E) = \delta(E).$$

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Thank you!